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A Simplified Approach for Bending Analysis of Beams Resting on a Two-Parameter Elastic Foundation

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Abstract. This paper presents a simplified method for analyzing beams interacting with two-parameter elastic foundations. The study develops and examines a refined mathematical model that captures the mechanical interaction between the beam and the elastic foundation, providing an accurate description of the beam's stress-strain state. The scientific novelty and practical significance lie in deriving analytical expressions for vertical displacements, bending moments, shear forces, and the distributions of normal and shear stresses along the beam. The methodology is based on theoretical analysis using an enhanced model of the two-parameter elastic foundation. As a result, analytical solutions are obtained, enabling evaluation of how the foundation's stiffness parameters influence structural behavior. The study establishes that the distributions of deflection and bending moment are symmetric about the midspan, whereas the distributions of shear forces and shear stresses are antisymmetric. Findings indicate that increasing foundation stiffness reduces deflections but maintains the overall pattern of internal force distribution. Particular attention is given to the case where $\alpha < 1$, which reveals specific characteristics of the beam's stress-strain behavior.

Keywords: beam, vertical displacements, deflections, bending moments, shear forces, modulus of elasticity, stress-strain state

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Introduction

The analysis of beams on elastic foundations is a classical problem in structural mechanics, and its rigorous formulation reduces to solving a contact interaction problem between a structure and its foundation. A beam resting on an elastic foundation represents a statically indeterminate system, since the equilibrium condition alone, equating the external load to the total reaction of the foundation, is insufficient to determine the distribution of this reaction along the beam length and to evaluate internal forces in its cross-sections [1-2]. The beam on elastic foundation model is widely used in the analysis of the interaction between structures and soil in geotechnical, road, and railway engineering, as well as in structural mechanics. The simplest and most widely used approach is the one-parameter model proposed by Winkler, in which the foundation is represented as a system of independent vertical elastic springs. Despite its simplicity and practical convenience, the Winkler model does not account for interaction between adjacent foundation elements, which leads to discontinuities in the displacement field and reduces the accuracy of calculations [3-5].

Subsequently, more advanced models of elastic foundations were developed, taking into account shear interaction and the physical properties of the soil, as well as calculation methods based on variational principles and the theory of elasticity. A significant contribution to the development of this field has been made by numerical methods, primarily the finite element method, which enables the analysis of nonlinear material behavior, contact phenomena, and variations in foundation stiffness. More advanced mechanical models, such as those proposed by Pasternak, Kerr, Vlasov, and Reissner, provide greater physical accuracy; however, their practical application is often limited by the complexity of determining model parameters [6-7]. Classical works by Vlasov, Leontiev, and their followers laid the theoretical foundations for the investigation of beams on elastic foundations, incorporating variational methods and analytical solutions. In most of these studies, the modulus of subgrade reaction was assumed to be constant along the length of the structure, which significantly simplifies the analysis but does not always reflect the actual heterogeneity of the foundation. In [8], the influence of symmetric heterogeneity of the foundation within the Winkler model on the stress-strain state of a beam was investigated. The subgrade modulus was defined as a quadratic function along the span, and the degree of heterogeneity was characterized by a specific parameter. The calculations were performed using the Ritz-Timoshenko variational method. Modern studies pay considerable attention to the analysis of one- and two-parameter elastic foundation models, in which, in addition to the subgrade reaction modulus, an additional parameter characterizing the shear stiffness of the soil is introduced. These include, in particular, the Pasternak model and models based on representing the foundation as an elastic layer of finite thickness [9]. In [10], a refined model of the contact layer was proposed for analyzing the stress-strain state of a multilayer beam. Unlike earlier studies, this model accounts for the nonlinear distribution of displacements across the thickness of the contact layer, ensuring correct satisfaction of boundary conditions, in particular, the zero shear stress condition at corner points. The foundation is considered as an elastic layer of finite thickness resting on a rigid base. In contrast to simplified approaches, this model does not neglect any components of stresses, strains, or displacements in the soil base [11]. In recent years, considerable attention has been devoted to the development of simplified yet sufficiently accurate models aimed at practical engineering applications. In this direction, combined analytical-numerical approaches, iterative methods, boundary element schemes, and modified finite element models have been proposed, providing good agreement with classical solutions and experimental data. A simplified analytical method based on a modified two-parameter foundation model exceeds the classical Winkler model in accuracy while being less complex than the Vlasov model [12-14].

In study [15], a simplified method for analyzing beams on a two-parameter elastic foundation was proposed. The foundation was characterized by two elastic parameters, and a simplified governing differential equation of beam bending was derived. The validity of the approach was verified by comparison with the classical Winkler and Vlasov models. The results showed good agreement with established elastic foundation theories, confirming the reliability of the proposed analytical method.

Modern research is focused on the development of universal numerical methods that can account for arbitrary beam and foundation models, nonlinear material behavior, contact discontinuities, and spatial variability of parameters, all without a significant increase in computational cost. In this context, a promising approach is based on the independent discretization of the beam and the foundation, followed by a reduction in the number of variables, which improves the accuracy and efficiency of the analysis while preserving universality and ease of implementation within the framework of standard finite element formulations [16-18]. In [19], a refined mathematical model of a beam on a two-parameter elastic foundation was developed, together with the governing equilibrium equation and its analytical solutions for various boundary and loading conditions. The study focused on deflections, bending moments, shear forces, and normal and shear stresses, with a comparison to the classical Winkler model. The results confirm that the proposed approach is reliable, practically applicable, and more accurate in capturing stress-strain behavior than classical models.

Based on the analysis of existing calculation methods, which are often characterized by increased complexity, as well as the limitations of the classical Winkler model, this paper proposes a new simplified method for analyzing a beam on a two-parameter elastic foundation. The developed model allows for the determination of vertical displacements, bending moments, and shear forces, as well as the evaluation of normal and shear stresses in beam cross-sections while accounting for interaction with the elastic foundation. An important advantage of the proposed method over existing models is its improved accuracy compared to the widely used Winkler model, along with reduced computational complexity relative to the Vlasov model, which ensures its practical applicability in engineering analysis.

The aim of this study is to develop and justify a simplified method for analyzing a beam on a two-parameter elastic foundation that improves the accuracy of determining deflections, internal forces, and normal and shear stresses in beam cross-sections compared to the Winkler and Vlasov models, while maintaining computational efficiency and simplicity of implementation.

The methodology

The elastic foundation of thickness h and modulus of elasticity E is modeled as an elastic half-space and is considered in a Cartesian coordinate system ($-\infty \leq x_1 \leq \infty$, $0 \leq x_3 \leq \infty$). A beam of length l, thickness h_0, and modulus of elasticity E is considered within the coordinate domain$\left(-\frac{l}{2} \leq x_1 \leq \frac{l}{2}\right)$, $\left(-\frac{h_0}{2} \leq x_3 \leq \frac{h_0}{2}\right)$. The beam rests on the surface of the elastic foundation and interacts with it along the contact interface (Fig. 1).

To determine the stress-strain state of the elastic foundation, the displacement function method is applied. In deriving the governing relations, the fundamental equations of two-dimensional elasticity theory are used, establishing the relationships between stresses, strains, and displacements in an elastic medium. Taking into account the modulus of elasticity of the foundation and the shear modulus of the beam material, the standard differential equation of elasticity theory describing the stress-strain state of the system is adopted:

$$\nabla^2\nabla^2 = \frac{\partial^4 F}{\partial x_1^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 F}{\partial x_3^4} = 0 \tag{1}$$

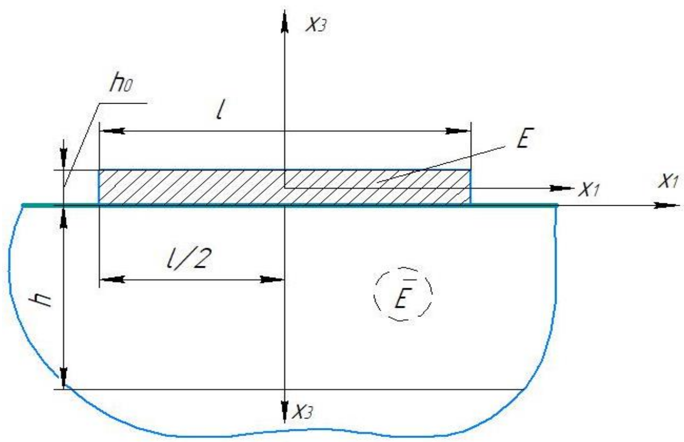

Figure 1. Beam on an elastic foundation

To obtain the solution of the biharmonic equation (1), the displacement function $F(x_1,x_3)$ is assumed in the following form:

$$F(x_1, x_3) = \delta(x_3) \cdot W(x_1) \tag{2}$$

where $\delta(x_3)$ – is the distribution function;

$W(x_1)$ – is the deflection function of the elastic foundation.

Using the displacement function (2) and the relations:

$$\frac{d^2W(x_1)}{dx_1^2} = -\bar{k}^2 W(x_1), \frac{d^4W(x_1)}{dx_1^4} = -\bar{k}^4 W(x_1),$$

The displacement and stress components are determined.

Displacement components:

$$U_1(x_1, x_3) = -h \cdot \varphi(z_0)\frac{dW(x_1)}{dx_1}$$

$$U_3(x_1, x_3) = f(z_0) \cdot \mathrm{W}(x_1)$$

$$\varphi(z_0) = \frac{(1+\nu)}{(1-\nu)}\delta'(z_0) \tag{3}$$

$$f(z_0) = \delta''(z_0) - \frac{2}{(1-\nu)}k^2\delta(z_0)$$

$$\delta(z_0) = (C_1 + C_2 z_0)e^{-kz_0}, z_0 = \frac{x_3}{h}, k^2 = \bar{k}^2 \cdot h^2$$

where ν – is Poisson's ratio;
z_0 – is the dimensionless transverse coordinate in the elastic foundation;
$\varphi(z_0)$, $f(z_0)$ – describe the distribution of displacements (U_1, U_3);
k – is the deformed state parameter, which depends on boundary conditions.
Stress components:

$$\sigma_1 = -\frac{\bar{E}h}{12} \cdot \psi'(z_0) \frac{d^2W(x_1)}{dx_1^2}$$

$$\tau_{13} = \frac{\bar{E}h^2}{12} \cdot \psi(z_0) \frac{d^3W(x_1)}{dx_1^3}$$

$$\sigma_3 = -\frac{\bar{E}h^3}{12} \cdot \alpha(z_0) \frac{d^4W(x_1)}{dx_1^4} \tag{4}$$

$$\psi(z_0) = 12\left[\delta(z_0) + \frac{\nu}{k^2}\delta'''(z_0)\right]$$
$$\alpha(z_0) = \frac{12}{k^4}[\delta'''(z_0) - (2+\nu)k^2\delta'(z_0)]$$

where $\psi'(z_0)$, $\psi(z_0)$, $\alpha(z_0)$ – are functions that characterize the voltage distribution σ_1, τ_{13}, σ_3.

When the displacement function (2) is inserted into equation (1), we get an expression of the following kind:

$$\delta(x_3) \cdot \frac{d^4W(x_1)}{dx_1^4} + 2\delta''(x_3) \cdot \frac{d^2W(x_1)}{dx_1^2} + \delta^{IV}(x_3) \cdot W(x_1) = 0$$

$$\left[\delta^{IV}(z_0) - 2 \cdot \bar{k}^2h^2\delta''(z_0) + \bar{k}^4h^4\delta(z_0)\right] \cdot W(x_1) = 0 \tag{5}$$

$$\delta^{IV}(z_0) - 2 \cdot k^2\delta''(z_0) + k^4\delta(z_0) = 0$$

To determine the unknown constants A_0, B_0, C_0, the following conditions are adopted:
Boundary conditions: $z = \frac{1}{2}$

$$\tau_{13}^0 = 0; \psi_0\left(\frac{1}{2}\right) = 0, A_0 - \frac{C_0}{2} + \frac{1}{8} = 0$$

$$\sigma_3^0 = q: Eh_0^3\delta_0\left(\frac{1}{2}\right)\frac{d^4W_0(x_1)}{dx_1^4} = q \tag{6}$$

$$\delta_0\left(\frac{1}{2}\right) = B_0 - \frac{A_0}{2} + \frac{C_0}{8} - \frac{1}{48}$$

where $q(x_1)$ – is the intensity of the external distributed load acting on the beam.
Under contact conditions, when the beam deflection coincides with the deflection of the elastic foundation surface$W_0(x_1) = W(x_1)$, a system of equations for determining the unknown

parameters is obtained:

$$1 = f(0), \qquad C_0 + \frac{1}{2} = \frac{h}{h_0}\varphi(0)$$

$$A_0 + \frac{C_0}{2} + \frac{1}{8} = \frac{1}{12}\frac{\bar{E}h^2}{Eh_0^2}\beta_0; \beta_0 = \psi(0) \tag{7}$$

$$B_0 + \frac{A_0}{2} + \frac{C_0}{8} + \frac{1}{48} = -\frac{1}{12}\frac{\bar{E}h^3}{Eh_0^3}\alpha_0; \alpha_0 = \alpha(0)$$

From the basic conditions using (7), the unknown constants are determined:

$$C_0 = \frac{1}{12}\frac{\bar{E}h^2}{Eh_0^2}\beta_0, \quad A_0 = -\frac{1}{8} + \frac{1}{24}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 \tag{8}$$

$$B_0 = \frac{1}{24} - \frac{1}{32}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 - \frac{1}{12}\frac{\bar{E}h^3}{Eh_0^3}\alpha_0$$

where β_0, α_0 are the values of the stress distribution functions (τ_{13}, σ_3) of the elastic foundation at the contact point.

Taking into account the functions obtained by the displacement function method, according to expression (3) and the values of their derivatives at the point *z₀=0*, the following equations are obtained from the first two expressions of conditions (7):

$$1 = \delta''(0) - \frac{2k^2}{1-\nu}\cdot\delta(0)$$

$$\frac{1}{12}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 + \frac{1}{2} = \frac{h}{h_0}\cdot\frac{1+\nu}{1-\nu}\delta'(0) \tag{9}$$

The general solution of this equation can be obtained using three different types of solutions:

$$\lambda^2 = k^2\left(1 \pm \sqrt{1-\alpha}\right); k_\omega^4 = \alpha\cdot(k^2)^2$$

$$1.\ \alpha = 1: \alpha(z_0) = (C_1 + C_2 z_0)e^{-kz_0} + (C_3 + C_4 z_0)e^{kz_0} \tag{10}$$

$$2.\ \alpha > 1: \delta(z_0) = [C_1\cos(\beta_1 z_0) + C_2\sin(\beta_1 z_0)]e^{-\alpha_1 z_0} + [C_3\cos(\beta_1 z_0) + C_4\sin(\beta_1 z_0)]e^{\alpha_1 z_0} \tag{11}$$

$$3.\ \alpha < 1: \delta(z_0) = C_1 e^{\alpha_2 z_0} + C_2 e^{-\alpha_2 z_0} + C_3 e^{\beta_2 z_0} + C_4 e^{-\beta_2 z_0} \tag{12}$$

where

$$\alpha_1 = k\sqrt{\frac{\alpha+1}{2}}, \quad \beta_1 = k\sqrt{\frac{\alpha-1}{2}}$$

$$\alpha_2 = k\sqrt{1-\sqrt{1-\alpha}}\,,\ \ \beta_2 = k\sqrt{1+\sqrt{1-\alpha}}$$

The general solution of system (10)-(12) is determined depending on the value of parameter α. Three different cases are considered, for each of which appropriate calculation methods are applied. Previous studies examined in detail the first case *α=1,* and the second *α>1*, for which analytical calculations and analysis of results were performed [13, 15, 19, 20].

In the present study, particular attention is paid to the third case corresponding to the condition *α<1*, for which the necessary computational investigations are carried out. When applying the general solution to an elastic foundation $(z_0 \to \infty,\ \delta(z_0)=0)$, it takes the following form:

$$\delta(z_0) = C_2 e^{-\alpha_2 z_0} + C_4 e^{-\beta_2 z_0} \tag{13}$$

Thus, when using the proposed method, if the functions $W(x_1)$ and $\delta(z_0)$ are given, the stress-strain state of the elastic foundation can be determined relatively simply.

If a distributed load acts normally $\sigma(x_1)$ and tangentially $\tau(x_1)$ on the elastic foundation ($z_0 = 0$), the boundary conditions (4) are written as follows:

$$\begin{aligned} \tau_{13} = \tau\colon \beta_0 \frac{\bar{E}h^2}{12}\frac{d^3W(x_1)}{dx_1^3} = \tau(x_1), \psi(0) = \beta_0 \\ \sigma_3 = \sigma\colon \alpha_0 \frac{\bar{E}h^3}{12}\frac{d^4W(x_1)}{dx_1^4} = \sigma(x_1), \alpha(0) = \alpha_0 \end{aligned} \tag{14}$$

where α_0, β_0 – are parameters of the normal and shear distributed loads.

In general form, for $\beta_0 \neq 0$, $\alpha_0 \neq 0$, boundary conditions (14) using expressions (4) are written as:

$$\begin{aligned} \psi(0) = \beta_0\colon \frac{\beta_0}{12} = \delta(0) + \frac{\nu}{k^2}\delta''(0) \\ \alpha(0) = \alpha_0\colon \frac{k^4}{12}\alpha_0 = \delta'''(0) - (2+\nu)k^2\delta'(0) \end{aligned} \tag{15}$$

Let us determine the values of the main function (13) and its derivatives at $z_0=0$.

$$\begin{aligned} \delta'(z_0) &= -\sqrt{2}C_4 k e^{-\sqrt{2}kz_0} \\ \delta''(z_0) &= 2C_4k^2e^{-\sqrt{2}kz_0} \\ \delta'''(z_0) &= -2\sqrt{2}C_4k^3e^{-\sqrt{2}kz_0} \end{aligned} \tag{16}$$

$$\delta(0) = C_2 + C_4,\ \delta'(0) = -\sqrt{2}C_4k,\ \ \delta''(0) = 2C_4k^2,\ \delta'''(0) = -2\sqrt{2}C_4k^3$$

Substituting (16) into (15), the following system is obtained:

$$C_2 = -\frac{\sqrt{2}k\alpha_0 - 2v\beta_0 + 2\sqrt{2}kv\alpha_0}{24v}, \qquad C_4 = \frac{\sqrt{2}k\alpha_0}{24v} \tag{17}$$

Substituting constants (17) into (13), the main function is obtained $\delta(z_0)$:

$$\delta(z_0) = -\frac{\sqrt{2}k\alpha_0 - 2v\beta_0 + 2\sqrt{2}kv\alpha_0}{24v} \cdot e^{-\alpha_2 z_0} + \frac{\sqrt{2}k\alpha_0}{24v} \cdot e^{-\beta_2 z_0} \tag{18}$$

Solving equations (9), the parameters of stress distribution functions are determined:

$$\alpha_{1k} = \frac{\sqrt{2}k^3}{12v} + \frac{\sqrt{2}k^3(1+2v)}{12v(1-v)} - \frac{\sqrt{2}k^3}{12v(1-v)} - \frac{2k^2}{(1-v)} \cdot \frac{Eh_0}{\bar{E}h} \cdot \frac{k^2(1+v)}{12v(1-v)}$$

$$\alpha_0 = \frac{1}{\alpha_{1k}} - \left[1 + \frac{k^2 Eh_0^2}{(1-v)\bar{E}h^2}\right] \tag{19}$$

$$\beta_{1k} = \frac{h}{h_0} \cdot \frac{k^2(1+v)}{12v(1-v)}\alpha_0 - \frac{1}{2} \qquad \beta_0 = \frac{12Eh_0^2}{\bar{E}h^2}\beta_{1k}$$

where v – is Poisson's ratio;

κ – is a dimensionless parameter of the stress-strain state.

Taking into account relation (8), from boundary condition (6), the governing differential equation for beam deflection is obtained:

$$\gamma = 1 - \frac{1}{2} \cdot \frac{\bar{E}h^2}{Eh_0^2}\beta_0 - \frac{\bar{E}h^3}{Eh_0^3}\alpha_0 \tag{20}$$

where J – is the moment of inertia of the beam cross-section.

The parameters of the bending stiffness of the beam, taking into account (3), (8), and (19), are written as:

$$g_2 = 12\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(A_0 - C_0 \cdot z + \frac{z^2}{2}\right)dz = -g_0$$

$$g_2 = 12\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(-\frac{1}{8} + \frac{1}{24}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 - \frac{1}{12}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 \cdot z + \frac{z^2}{2}\right)dz = -g_0 \tag{21}$$

$$g_0 = 1 - 6\beta_{1k}, \qquad z = \frac{x_3}{h_0}$$

where β_{1k} – is the parameter of the elastic foundation.

z – is the dimensionless transverse coordinate along the height of the beam.

Detailed derivations of the expressions for the bending moment M, shear force Q, and the normal σ_1^0 and shear stresses τ_{13}^0 of a beam on a two-parameter elastic foundation can be found in our previous publications [15,19]. Therefore, only the final governing relations adopted in the present study are given below.

$$\sigma_1^0 = h_0 \cdot \varphi_0(z)\frac{M}{J}, \quad \tau_{13}^0 = -h_0^2 \cdot \psi_0(z)\frac{Q}{g_0 J}$$
$$M = -EJ\frac{d^2 W_0(x_1)}{dx_1^2}, \quad Q = -EJg_0\frac{d^3 W_0(x_1)}{dx_1^3} \tag{22}$$

Findings/Discussion

This study considers a beam interacting with a two-parameter elastic foundation. The beam is subjected to a uniformly distributed load $q=1kN/m$. The geometric parameters of the beam are as follows: length $l=10m$, height $h_0=0.25m$, and modulus of elasticity of the material $E=1\cdot10^3Pa$.

Table 1. Values of vertical displacements for foundation elastic modulus $\bar{E} = 1Pa, \bar{E} = 10Pa, \bar{E} = 100Pa$

x	$W_1(x)$, $\bar{E} = 1$Па	$W_2(x)$, $\bar{E} = 10$Па	$W_3(x)$, $\bar{E} = 100$Па
0	0	0	0
1	10.391	5.373	0.109
2	19.659	10.166	0.207
3	26.914	13.918	0.283
4	31.521	16.3	0.331
5	33.1	17.116	0.348
6	31.521	16.3	0.331
7	26.914	13.918	0.283
8	19.659	10.166	0.207
9	10.391	5.373	0.109
10	0	0	0

The main parameters of the elastic foundation: $\bar{E} = 1Pa, \bar{E} = 10Pa, \bar{E} = 100Pa, \ \nu = 0.25 and h = 1m, \ \alpha = 0$. The adopted parameters make it possible to evaluate the stress-strain state of the beam on the elastic foundation and the influence of foundation characteristics on the distribution of internal forces and deflections. The performed analytical and numerical study provided the distributions of deflections, bending moments, shear forces, and stresses in the beam. It was shown that for $\alpha<1$, the stress-strain state of the beam changes significantly due to the specific features of beam-foundation interaction.

The results of the vertical displacement analysis are presented in Table 1 and Figure 2. The deflection distribution *W(x)* is symmetric with respect to the midspan, with maximum values occurring at the beam center ($x=l/2$). The analysis shows that increasing the elastic modulus of the foundation $\bar{E}$ significantly reduces beam deflections along the entire span. The largest displacements correspond to $\bar{E} = 1Pa$, whereas increasing the foundation stiffness to $\bar{E} = 100Pa$ results in a considerable reduction in deflections.

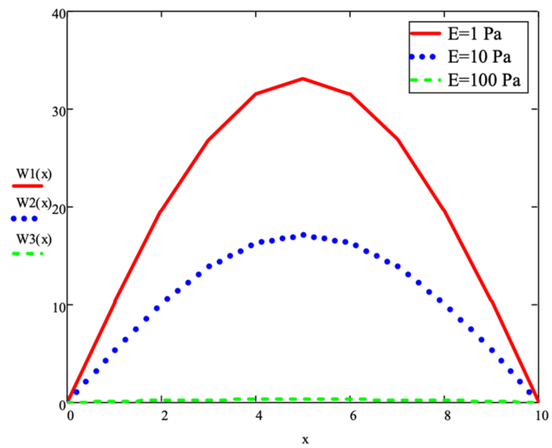

Figure 2. Distribution of vertical displacements *W(x)* for different values of the foundation elastic modulus $\bar{E}$.

The results of the calculation of the bending moment *M(x)* and shear force *Q(x)* are presented in Table 2, while their distribution along the beam span is shown in Figure 3, respectively. As follows from the data in Table 2 and the corresponding graphical dependencies, the distribution of the bending moment *M(x)* is symmetric with respect to the midspan. The maximum bending moment occurs in the central part of the beam *(x=l/2)*, whereas it reduces to zero at the supports, which is consistent with classical beam bending theory. In contrast, the distribution of the shear force *Q(x)* exhibits an antisymmetric pattern: the maximum absolute values are observed in the regions near the supports, while at the midspan the shear force becomes zero. When passing through the midspan, the sign of the shear force changes, reflecting the variation in the internal force distribution within the beam.

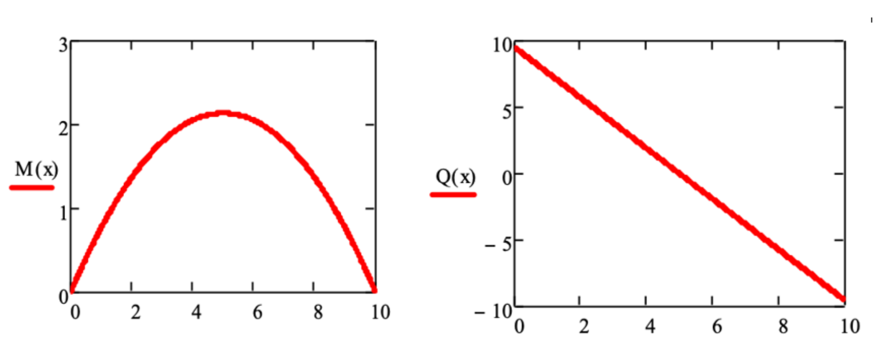

Figure 3. Distribution of the bending moment *M(x)* and shear force *Q(x)* along the beam length

The results of the calculation of normal and shear stresses in the beam on an elastic foundation are presented in Table 3 and Figures 4-5.

Table 2. Bending moment *M(x)* and shear force *Q(x)* values

x	*M(x)*	*Q(x)*
0	0	9.528
1	0.77	7.622
2	1.369	5.717
3	1.797	3.811
4	2.054	1.906
5	2.14	0
6	2.054	-1.906
7	1.797	-3.811
8	1.369	-5.717
9	0.77	-7.622
10	0	-9.528

The distribution of normal stresses $\sigma_{10}(x,z)$ is symmetric with respect to the midspan, with maximum values observed in the central part of the beam, corresponding to the region of maximum bending moment.

Table 3. Variation of normal stresses $\sigma_{10}(x,z)$ along the beam length

x	$\sigma_{10}(x,-\frac{1}{2})$	$\sigma_{10}(x,0)$	$\sigma_{10}(x,\frac{1}{2})$	$\sigma_{10}(x,1)$
0	0	0	0	0
1	175.816	249.759	323.702	397.645
2	312.562	444.016	575.47	706.924
3	410.237	582.771	755.305	927.838
4	468.842	666.024	863.205	$1.06 \cdot 10^3$
5	488.378	693.775	899.172	$1.105 \cdot 10^3$
6	468.842	666.024	863.205	$1.06 \cdot 10^3$
7	410.237	582.771	755.305	927.838
8	312.562	444.016	575.47	706.924
9	175.816	249.759	323.702	397.645
10	0	0	0	0

The shear stresses $\tau(x,z)$ vary both along the beam length and across the thickness of the cross-section, vanishing at the boundaries and reaching maximum values near the neutral axis. The obtained stress distributions are consistent with the classical behavior of beams on elastic foundations.

The obtained analytical and numerical results provide a comprehensive analysis of the stress-strain state of a beam on a two-parameter elastic foundation. The developed method demonstrates reliable results consistent with the classical theory of beams on elastic foundations.

The deflection distribution $W(x)$ is symmetric with respect to the midspan, with maximum values occurring in the central region of the beam

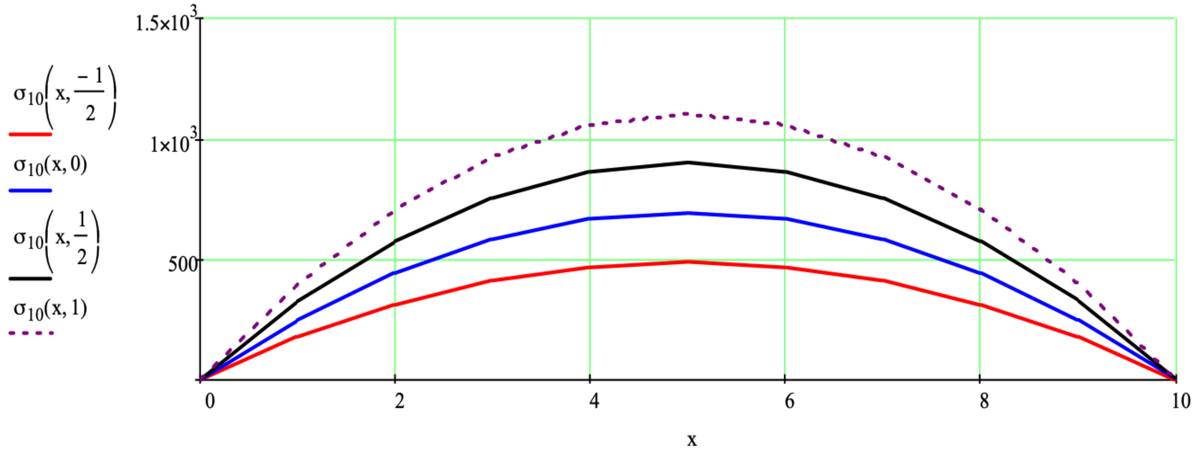

Figure 4. Distribution of normal stresses $\sigma_{10}(x,z)$

An increase in the elastic modulus of the foundation E leads to a significant reduction in beam deflections. The bending moment distribution is symmetric with a maximum at the midspan, whereas the shear force distribution is antisymmetric and becomes zero at the center of the beam. The stress analysis showed that normal stresses $\sigma_{10}(x,z)$ are symmetrically distributed and attain maximum values in the region of maximum bending moment. Shear stresses $\tau(x,z)$ exhibit a nonuniform distribution across the beam length and cross-section thickness, vanishing at the boundaries and reaching extreme values near the neutral axis. Calculations for the case $\alpha<1$ revealed specific features of the beam-foundation interaction and demonstrated a significant influence of foundation parameters on the stress-strain state of the beam. The proposed simplified model provides reliable results with relatively simple implementation.

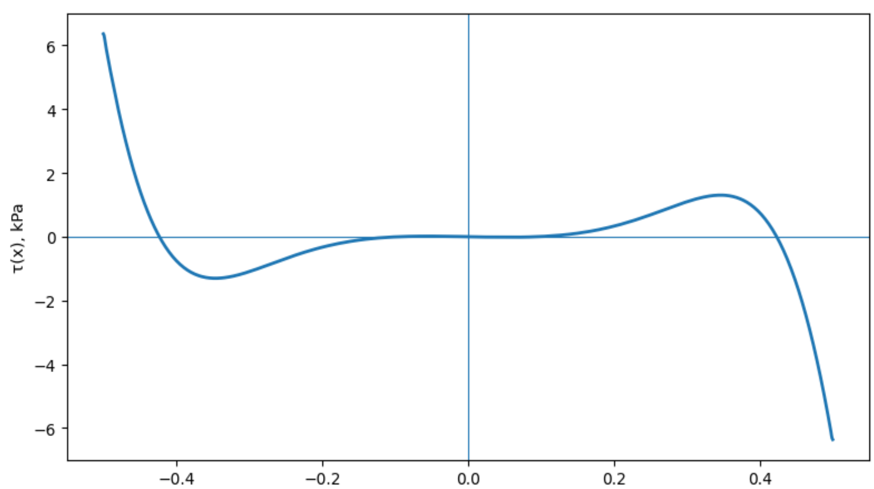

Figure 5. Distribution of shear stresses $\tau(x,z)$

The graphical dependencies obtained using Mathcad clearly illustrate the behavior of the investigated quantities and confirm the reliability of the obtained results. The presented distributions are consistent with the behavior of bending elements on an elastic foundation and can be used for further analysis and practical design calculations.

Conclusion

A simplified method for analyzing a beam on a two-parameter elastic foundation has been developed, allowing the interaction between the beam and the elastic foundation to be taken into account. As a result of the performed calculations, analytical relationships were obtained for determining vertical displacements, bending moments, shear forces, as well as normal and shear stresses in the beam. It was established that the distributions of deflections and bending moments are symmetric with respect to the midspan, whereas the distributions of shear forces and shear stresses are antisymmetric, which is consistent with the classical behavior of bending elements. The analysis of the obtained results showed that an increase in the stiffness of the elastic foundation leads to a reduction in deflections along the entire beam length. At the same time, the general pattern of displacement and internal force distribution is preserved, and the minimum displacement values are observed at the highest values of the foundation elastic modulus. Particular attention was given to the case $\alpha<1$, in which the specific features of the interaction between the beam and the elastic foundation are most pronounced. In this case, a more significant change in the stress–strain state of the structure is observed. The obtained results may serve as a basis for the further development of refined models describing the interaction between structures and elastic foundations.

The contribution of the authors

S. Akhazhanov – drafting, methodology, modeling
A. Kassenova - data collection, analysis
B. Nurlanova - analysis, interpretation
N. Medeubaev – data collection, article design.

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